International Journal of Theoretical Physics, Vol. 7, No. 3 (1973), pp. 213–228

Further Studies in Aesthetic Field Theory; Existence of a Bounded Particle

M. MURASKIN

University of North Dakota, Grand Forks, North Dakota

Received: 25 January 1972

Abstract

We have looked into various possibilities within aesthetic field theory, giving particular attention to the case of $g = 0$. The $g = 0$ situation can be associated with the introduction of Newtonian absolute time into aesthetic field theory. It can be argued that Lorentz invariant boundary conditions for the universe are unlikely, giving impetus to the study of $g = 0$. We find that the field equations had to be modified from the form that they take when $g = 0$. Also, an infinite number of integrability equations have to be satisfied. We have required that our data have an underlying structure that is invariant under $O(3) \times T$. This set of data appeared satisfactory with respect to integrability and gave rise to a minimum in g_{00} at the origin. After a long computer run along the coordinate axes, we also found a bound on our particle-like object. This is the first time we have been able to obtain such a result.

1. *Introduction*

In a series of papers (Muraskin, 1970, 1971a, 1971b, 1972a, 1972b, 1972c; Muraskin & Clark, 1970; Muraskin & Ring, 1972a, 1972b), we have been studying a field theory based on mathematically aesthetic principles. We introduced a change function and then required that the change function determine its own change in the same manner that it causes the change of other tensor functions. This led to the field equations for the change function

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} = \Gamma^i_{mk}\Gamma^m_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^m_{jk}\Gamma^i_{ml} \qquad (1.1)$$

However, within the framework of such a field theory there still remains various possibilities to consider. For example, we previously pointed out that the integrability equations can be most simply satisfied by the condition $R^i_{jkl} = 0$. But in a separate paper (Muraskin, 1972c), we proved local existence also when $R^i_{jkl} \neq 0$. Now, if $R^i_{jkl} = 0$ at one point, it follows that $R^i_{jkl} = 0$ at all points, as a consequence of the field equations. Thus, the $R^i_{jkl} = 0$ and $R^i_{jkl} \neq 0$ theories are distinct. Some further criterion would be needed to decide between these two possibilities. At present we cannot say with any assurance which situation is, *a priori*, more reasonable.

In this paper, we shall study some other possible freedoms consistent with the notion that the change function determines its own change.

As for results—working with a situation where $g = 0$, we have made a long-time computer run along the coordinate axes and found a bound on a particle-like object. This is the first time we have come across such results.

2. *The Determinant of* g_{ij}

The second-rank symmetric tensor g_{ij} obeys the equation

$$\frac{\partial g_{ij}}{\partial x^k} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{il} = 0 \tag{2.1}$$

This equation can be obtained by introducing four basis vectors

$$g_{ij} = e^\alpha{}_i e^\beta{}_j g_{\alpha\beta} \tag{2.2}$$

Up to now, we have always chosen $g_{\alpha\beta}$ to be the Minkowski metric $(-1,-1,-1,1)$, and thus, the determinant of g is always negative. However, the choice of the sign of g is not so cut and dried and should be open to discussion. We note, as a consequence of the field equations, the sign of g is preserved at all points of space-time.

Lanzcos (1966) has introduced $g > 0$ some time ago, arguing that such a choice does not exclude wave motion. In our previous work (Muraskin, 1972a) we found in a perturbation calculation of equation (1.1), that wave motion appears in first order. This result was independent of whether g was negative or positive. Thus, we should keep in mind the possibility of determinant positive. All the basic equations of the theory remain unchanged in this case. Thus far, the difficulties with $g > 0$ are that it is very difficult to find solutions of the integrability equations together with the infinite restrictions that all invariants be zero at the origin. This latter condition is necessary so that the natural boundary conditions $\Gamma^i_{jk} \to 0$ at infinity be satisfied. At present, we have no proof that these conditions cannot all be satisfied.

What about determinant zero? This situation is more complicated since g^{ij} is now singular. This implies that it is necessary to rederive the field equations, which we shall do later on. The field equation in this case will differ from (1.1).

A reasonable boundary condition is $e^\alpha{}_i \to \delta^\alpha{}_i$ at infinity, and thus $g_{ij} \to g_{\alpha\beta}$ there. Another possibility is $e^\alpha{}_i \to 0$ and thus $g_{ij} \to 0$ at infinity.

By requiring $g_{ij} \to g_{\alpha\beta}$ at infinity, the choice of $g_{\alpha\beta}$ can be related to the invariance group at infinity. $g_{\alpha\beta} = (-1,-1,-1,+1)$ is invariant under Lorentz transformations. $g_{\alpha\beta} = (1,1,1,1)$ is invariant under four dimensional rotations. Finally, $g_{\alpha\beta} = (1,1,1,0)$ which has determinant zero, is invariant under $O'(3) \times T$, where $O'(3)$ describes three dimensional rotations† and T refers to time translation. We note $g_{\alpha\beta} = (1,1,1,0)$ appears

† The prime denotes that the rotations are inhomogeneous.

also in Anderson (1967) and Trautmann (1964) in the discussion of Newtonian physics using the objects g_{ij}, Γ^i_{jk}. In the $g = 0$ situation, $g_{\alpha\beta}$ is also invariant under a wider group. However, as in Anderson (page 110) we may also require that the unit vector (0,0,0,1) be invariant under the transformation group. This restricts us to $O'(3) \times T$.

Thus, the choice of $g_{\alpha\beta}$ may be related to the invariance group at infinity. In our previous papers, we have exclusively considered the case $g_{\alpha\beta} = (-1,-1,-1,1)$. However, it could be argued, from a fundamental point of view, that there may be an inherent weakness in requiring that the laws of physics be the same for all inertial systems. That is, the universe has boundary conditions which may be argued to be non-Lorentz invariant. An alternative hypothesis is that the boundary conditions should lead to a Newtonian absolute time. The invariance group would be $O'(3) \times T$ as defined by the invariance of $g_{\alpha\beta} = (1,1,1,0)$ and the time-like unit vector (0,0,0,1).

We feel that the case of $g = 0$ is worthy of additional study.

Although we have emphasized $g_{ij} \to g_{\alpha\beta}$ above, we do not mean to draw any definitive conclusions at this point in regard to the boundary conditions. The situation $g_{ij} \to 0$ at infinity is still a possibility.

3. $g = 0$ *Theory*

In this section we shall obtain the equations in $g = 0$ theory.

In equations (1.1) and (2.1), we note the presence of both covariant and contravariant indices. This situation would be what one would expect if g_{ij} has an inverse. That is, from

$$A_i = g_{ij} A^j \tag{3.1}$$

we can define A^j. Suppose we are dealing with the case of $g = 0$. Consider, for example, $g_{ij} = (1,1,1,0)$ at a point. Then in (3.1), when $i = 0$, we get $A^0 = \infty$ which is not an allowed field. We could, if we want to, introduce independent fields A'^i, A_i, g'_{ij}, g_{ij} etc. But, this is not the simplest thing to do. What we shall do is to only work with tensor quantities having subscript indices.

In the system where $g = 0$, for the change of a vector field, we write

$$dA_i = \Lambda_{mik} A_m dx_k \tag{3.2}$$

Note, the order of indices in the change function Λ_{mik} is of no significance. That is, if we had written instead

$$dA_i = \Theta_{imk} A_m dx_k \tag{3.3}$$

we could define $\Lambda'_{mik} \equiv \Theta_{imk}$ and then get back to structure (3.2). What we call the change function (Λ_{mik} or Λ'_{mik}) is of no importance. It will be determined by the same field equations.

In (3.2) the summation over repeated indices is understood from this example; $A_m B_m = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_0 B_0$. Thus, we are effectively introducing a metric (1, 1, 1, 1) which defines scalar products. It will be assumed, for the present, that this metric, as well as the concept of dimension, are not dynamical quantities.

The change of g_{ij} is given by

$$\frac{\partial g_{ij}}{\partial x_k} = A_{mik} g_{mj} + A_{mjk} g_{im} \tag{3.4}$$

The change function determines its own change according to

$$\frac{\partial A_{ijk}}{\partial x_l} = A_{mjk} A_{mil} + A_{imk} A_{mjl} + A_{ijm} A_{mkl} \tag{3.5}$$

Consider a function of the field variables that has a contraction in some index, for example $B_i \equiv g_{ij} A_j$. Then using (3.2) and (3.4) we see that

$$dB_i \neq A_{mik} B_m \, dx_k \tag{3.6}$$

Thus, it appears that not all vectors are treated in the same way, so far as their change is concerned. This might appear then to mean that the $g = 0$ theory is less 'aesthetic' than the $g \neq 0$ situation. However, on closer scrutiny, this need not be considered the case. Let us return to the $g \neq 0$ case. We can define $B_m = A_l g^{lj(0)} A_j$ where $g^{(0)}_{ij}$ is a non-dynamical metric, such as (1, 1, 1, 1). Then we get $dB_i \neq \Gamma^j_{ik} B_j dx^k$. Thus, the conclusion is that objects involving non-dynamical quantities do not obey the equations

$$\frac{\partial T^{ij\ldots}_{km\ldots}}{\partial x^l} + T^{tj\ldots}_{km\ldots} \Gamma^i_{tl} + T^{it\ldots}_{km\ldots} \Gamma^j_{tl} - T^{ij\ldots}_{tm\ldots} \Gamma^t_{kl} - T^{ij\ldots}_{kt\ldots} \Gamma^t_{ml} + \cdots$$
$$\equiv T^{ij\ldots}_{km\ldots;l} = 0 \tag{3.6}$$

That is, $T^{ij\ldots}_{km\ldots}$ is a function of Γ^i_{jk}, g_{ij} but not of $g^{(0)}_{ij}$. This situation is no different in $g = 0$ theory. The expression $B_i = g_{ij} A_j$ in $g = 0$ theory means $B_i = g_{ij} g^{jm(0)} A_m$ and thus, it involves a non-dynamical quantity. Thus, we do not expect a relationship of the type (3.2) to hold for this B_i. All tensor quantities of the dynamical fields g_{ij}, A_{ijk}, A_i behave in a uniform manner so far as their change is concerned. From this follows equations (3.2), (3.4) and (3.5).

When $g \neq 0$ we had that all scalars were constant. In the $g = 0$ case, we have to be more careful. For example, we have

$$\frac{\partial A_i A_i}{\partial x_m} = 2 A_i A_t A_{tim} \tag{3.7}$$

This is not, in general, zero. We could get all such quantities to be zero if

$$A_{tik} = -A_{itk}. \tag{3.8}$$

We can see this, in general, by considering

$$\frac{\partial A_{ij\ldots i\ldots j}}{\partial x_k} = A_{mik} A_{mj\ldots i\ldots j\ldots} + A_{mik} A_{ij\ldots m\ldots j\ldots} + \cdots \tag{3.9}$$

Interchanging the dummy indices m and i in the first term on the right and using (3.8), we see we get no contribution from contracted indices in $\partial A_{ij\ldots i\ldots j}/\partial x_k$. Note also, that if (3.8) is satisfied, expressions like $B_i = g_{ij} A_j$ will obey equations of the type (3.2).

At any rate, the result (3.7) is not unexpected. In $g \neq 0$ theory, if we consider quantities with a non-dynamical metric such as $A_i A_j g^{ij(0)}$, we find its change is not zero either.

In order for the mixed derivatives of A_l to be symmetric, it is necessary that

$$A_m B_{mlpk} = 0 \tag{3.10}$$

where

$$B_{tlmk} \equiv A_{tlj} A_{jkm} - A_{tlj} A_{jmk} + A_{jlk} A_{jtm} - A_{jlm} A_{jtk} \tag{3.11}$$

The mixed derivatives of g_{ij} are symmetric if

$$g_{mj} B_{mlpk} + g_{lm} B_{mjpk} = 0 \tag{3.12}$$

The mixed derivatives of A_{ijk} are symmetric if

$$A_{mjk} B_{mlpl} + A_{imk} B_{mjpl} + A_{ijm} B_{mkpl} = 0 \tag{3.13}$$

We note that the mixed derivatives of products of A_i, g_{ij}, A_{ijk} are symmetric if (3.10), (3.12) and (3.13) are satisfied.

If we find a simple set of data that satisfies these equations we can generate a more complicated set of data by using

$$A_{ijk} = e_{\alpha i} e_{\beta j} e_{\gamma k} A_{\alpha\beta\gamma} \tag{3.14}$$

where $A_{\alpha\beta\gamma}$ are the simple set of data and $e_{\alpha i}$ represents a general four-dimensional orthogonal matrix.†

The consistency equations are most simply satisfied if we take

$$B_{tlmk} = 0 \tag{3.15}$$

At this point we notice a complication that is absent in $g \neq 0$ theory. We can no longer show that if (3.15) is satisfied at one point, then it is satisfied at all points. If we compute the derivative of B_{tlmk} we get, on using the field equations

$$\frac{\partial B_{tlmk}}{\partial x_l} = A_{stl} B_{slmk} + A_{sll} B_{tsmk} + A_{sml} B_{tlsk} + A_{skl} B_{tlms} +$$
$$(A_{sjl} + A_{jsl})(A_{skm} A_{tlj} - A_{smk} A_{tlj} + A_{stm} A_{jlk} - A_{stk} A_{jlm}) \tag{3.16}$$

Making use of (3.15) at the origin is not sufficient to make (3.16) vanish as well. (Although we note (3.16) does vanish if (3.8) is satisfied.‡) Higher derivatives of B_{tlmk} would not be expected to be zero either, in general. Thus, the $g = 0$ theory is in danger of not leading to a consistent solution. However, there is a possibility, although one would think rather remote,

† If $e_{\alpha i} \to \delta_{\alpha i}$ at infinity, $A_{\alpha\beta\gamma}$ is a function of x so that $A_{ijk} \to 0$ at infinity.

‡ Equation (3.5) then, has a similar formal structure as (1.1).

that there exists a set of data at the origin such that all of the derivatives of B_{tlmk} just happen to vanish. Then the $g = 0$ theory would not have to be discarded.

We have been working with a set of $A_{\alpha\beta\gamma}$ data with attractive group theoretical properties. $A_{\alpha\beta\gamma}$ is required to be invariant under an $O'(3) \times T$ transformation. Our set of data has some remarkable properties. For a choice of parameters, this set of data satisfies (3.15). Then, without any further adjustments, we find (3.16) is satisfied as well. We have not been able to prove that all derivatives of B_{tlmk} are zero at the origin analytically, as there are an infinite number of conditions to be satisfied. However, what we have done is to make a run of 300 points down the x-axis (at ·0003 grid size). Then we calculated B_{tlmk} numerically at this point using the computer. We found that B_{tlmk} was zero to twelve decimal places. This was the same order of accuracy for which B_{tlmk} was zero at the origin. Furthermore, we ran 90 points down the x-axis with grid ·0003 and 90 points down the y-axis with grid ·0004 and compared this situation where the y-axis run was first and the x-axis run was second. We got agreement for all A_{ijk} and g_{ij} to twelve decimal places. These tests were also repeated periodically during our long runs. We found B_{tlmk} was zero in all cases again to the order of twelve decimal places.† Thus, these results strongly imply that our group theoretical data does, in fact, lead to an acceptable $g = 0$ theory, so far as integrability is concerned even though we have not been able to give a rigorous proof of local existence.

To further investigate this problem of integrability, we consider another set of $A_{\alpha\beta\gamma}$ data given by

$$
\begin{array}{llll}
A_{111} = -1 & A_{112} = +1 & A_{113} = +1 & A_{110} = -1 \\
A_{121} = +1 & A_{122} = -1 & A_{123} = -1 & A_{120} = +1 \\
A_{131} = -1 & A_{132} = +1 & A_{133} = +1 & A_{130} = -1 \\
A_{101} = +1 & A_{102} = -1 & A_{103} = -1 & A_{100} = +1 \\
A_{211} = -1 & A_{212} = +1 & A_{213} = +1 & A_{210} = -1 \\
A_{221} = +1 & A_{222} = -1 & A_{223} = -1 & A_{220} = +1 \\
A_{231} = -1 & A_{232} = +1 & A_{233} = +1 & A_{230} = -1 \\
A_{201} = +1 & A_{202} = -1 & A_{203} = -1 & A_{200} = +1 \\
A_{311} = -1 & A_{312} = +1 & A_{313} = -1 & A_{310} = +1 \\
A_{321} = +1 & A_{322} = -1 & A_{323} = +1 & A_{320} = -1 \\
A_{331} = +1 & A_{332} = -1 & A_{333} = +1 & A_{330} = -1 \\
A_{301} = -1 & A_{302} = +1 & A_{303} = -1 & A_{300} = +1 \\
A_{011} = -1 & A_{012} = +1 & A_{013} = -1 & A_{010} = +1 \\
A_{021} = +1 & A_{022} = -1 & A_{023} = +1 & A_{020} = -1 \\
A_{031} = +1 & A_{032} = -1 & A_{033} = +1 & A_{030} = -1 \\
A_{001} = -1 & A_{002} = +1 & A_{003} = -1 & A_{000} = +1
\end{array}
\qquad (3.17)
$$

This set of data obeys (3.15) but not (3.16). Another set of data that satisfies (3.15) but not (3.16) has the following non-vanishing $A_{\alpha\beta\gamma}$; A_{123}, A_{120},

† Henceforth, we shall call this sort of accuracy computer accuracy.

A_{213}, A_{210}, A_{333}, A_{000}. Furthermore, after an $e_{\alpha i}$ transformation and a run of 300 points down the x-axis, (3.13) was not satisfied by either data. Thus, the conclusion is that, in general, we cannot expect integrability to be satisfied in $g=0$ theory even when (3.15) is satisfied at the origin.

We may next ask whether there are other sets of $A_{\alpha\beta\gamma}$ other than our group theoretical data for which (3.16) holds and $B_{ilmk}=0$ is preserved away from the origin with computer accuracy. The answer is that there are. An example is the following.

$A_{111}=\phi_1$	$A_{112}=\phi_2$	$A_{113}=\phi_3$	$A_{110}=\phi_0$
$A_{121}=\phi_2$	$A_{122}=\phi_1$	$A_{123}=\phi_0$	$A_{120}=\phi_3$
$A_{131}=\phi_3$	$A_{132}=\phi_0$	$A_{133}=\phi_1$	$A_{130}=\phi_2$
$A_{101}=\phi_0$	$A_{102}=\phi_3$	$A_{103}=\phi_2$	$A_{100}=\phi_1$
$A_{211}=\phi_2$	$A_{212}=\phi_1$	$A_{213}=\phi_0$	$A_{210}=\phi_3$
$A_{221}=\phi_1$	$A_{222}=\phi_2$	$A_{223}=\phi_3$	$A_{220}=\phi_0$
$A_{231}=\phi_0$	$A_{232}=\phi_3$	$A_{233}=\phi_2$	$A_{230}=\phi_1$
$A_{201}=\phi_3$	$A_{202}=\phi_0$	$A_{203}=\phi_1$	$A_{200}=\phi_2$
$A_{311}=\phi_3$	$A_{312}=\phi_0$	$A_{313}=\phi_1$	$A_{310}=\phi_2$
$A_{321}=\phi_0$	$A_{322}=\phi_3$	$A_{323}=\phi_2$	$A_{320}=\phi_1$
$A_{331}=\phi_1$	$A_{332}=\phi_2$	$A_{333}=\phi_3$	$A_{330}=\phi_0$
$A_{301}=\phi_2$	$A_{302}=\phi_1$	$A_{303}=\phi_0$	$A_{300}=\phi_3$
$A_{011}=\phi_0$	$A_{012}=\phi_3$	$A_{013}=\phi_2$	$A_{010}=\phi_1$
$A_{021}=\phi_3$	$A_{022}=\phi_0$	$A_{023}=\phi_1$	$A_{023}=\phi_2$
$A_{031}=\phi_2$	$A_{032}=\phi_1$	$A_{033}=\phi_0$	$A_{030}=\phi_3$
$A_{001}=\phi_1$	$A_{002}=\phi_2$	$A_{003}=\phi_3$	$A_{000}=\phi_0$

This data obeys the integrability equations to computer accuracy. However, there is an obvious difficulty with this data. We note $A_{\alpha\beta\gamma}$ is symmetric in all indices. This symmetry is preserved by an $e_{\alpha i}$ transformation and by the field equations. Now, if we write out the field equations for $\partial A_{111}/\partial x_1$, we see, using the symmetry of A_{ijk}, that the right-hand side is made up of terms which are all squares. Thus, A_{111} will continue to increase monotonically as we move down the positive axis, leading to a singularity. Thus, not all data satisfying the integrability equations (to computer accuracy) will be acceptable. With our group theoretical data, it is not obvious at the outset, one way or another, whether a singularity will develop. We will have to make long runs away from the origin to see what the trends are.

As in our previous work, we seek a minimum (maximum) in g_{00}. At the origin, for simplicity we will work with $g_{ij}=(1,1,1,0)$.† Using (3.4) we get ($a=1,2,3$)

$$\frac{\partial g_{00}}{\partial x_a}=0 \tag{3.18}$$

† Strictly speaking, we should use (7.9) rather than $g_{ij}=(1,1,1,0)$ at the origin. However, it is much simpler to work with $g_{ij}=(1,1,1,0)$, and we shall do so here. Note, A_{ijk} are not affected by our choice of the value of g_{ij} at the origin.

Thus, g_{00} is automatically an extremum if we use $g_{ij} = (1, 1, 1, 0)$ (note, $\partial g_{00}/\partial x_0$ comes out to be zero as well). From the field equations (3.4) and (3.5), we get at the origin

$$2A_{ab} \equiv \frac{\partial^2 g_{00}}{\partial x_a \partial x_b} = 2(A_{10a}A_{10b} + A_{20a}A_{20b} + A_{30a}A_{30b}) \tag{3.19}$$

The conditions for a minimum are

$$A_{11} + A_{22} + A_{33} > 0 \tag{3.20a}$$

$$A_{11}A_{22} - (A_{12})^2 + A_{22}A_{33} - (A_{13})^2 + A_{33}A_{11} - (A_{13})^2 > 0 \tag{3.20b}$$

$$\det A_{ab} > 0 \tag{3.20c}$$

We define

$$\begin{aligned} a_a &\equiv A_{10a} \\ b_a &\equiv A_{20a} \\ c_a &\equiv A_{30a} \end{aligned} \tag{3.21}$$

Using (3.19) and (3.21) we see that (3.20a) is satisfied if a_a, b_a, c_a are not all zero. (3.20b) and (3.20c) become

$$\begin{aligned} &(a_1 b_2 - a_2 b_1)^2 + (a_3 b_1 - b_3 a_1)^2 + (a_1 c_2 - c_2 a_1)^2 + (a_1 c_2 - a_2 c_1)^2 \\ &+ (b_2 c_3 - b_3 c_2)^2 + (b_1 c_3 - b_3 c_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (a_2 c_3 - c_2 a_3)^2 + \\ &\qquad + (a_2 b_3 - b_3 a_2)^2 > 0 \end{aligned} \tag{3.22a}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \tag{3.22b}$$

Thus, if **a**, **b**, **c** are not multiples of one another, then we will have a minimum at the origin for g_{00}. g_{00} can never be a maximum once we take $g_{ij} = (1, 1, 1, 0)$ at the origin since (3.20a) is the sum of squares and is always positive. The significance of this result is not clear. It is true that in gravitational theory one can have only one sign of mass. But it is certainly premature to draw any conclusion at this stage in our case.

Note, it is not so simple a matter to obtain a minimum for g_{00}. For example, consider the data given by (3.17) as well as the case when A_{123}, A_{213}, A_{210}, A_{120}, A_{333}, A_{000} are the only non-zero $A_{\alpha\beta\gamma}$. Both cases give rise to (after an $e_{\alpha i}$ transformation in each case)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \tag{3.23}$$

to twelve decimal places at least. Thus, we get no minimum in g_{00} at the origin here.

We shall find, on the other hand, that our group theoretical data does give rise to a minimum in g_{00} at the origin.

4. *Group Theoretical Data*

At this point we display our group theoretical data

$$A_{\alpha\beta\gamma} = g_{\alpha\beta}\phi_\gamma + g_{\alpha\gamma}\theta_\beta + g_{\beta\gamma}\psi_\alpha + B_\chi \varepsilon_{\chi\alpha\beta\gamma} + \mu_\alpha\mu_\beta\mu_\gamma \tag{4.1}$$

with

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

and

$$\phi_a = \theta_a = \psi_a = B_a = \mu_a = 0 \qquad a = 1, 2, 3 \tag{4.3}$$

$\varepsilon_{\chi\alpha\beta\gamma}$ is the antisymmetric tensor. We assume the structure (4.1), (4.2) and (4.3) exists at all points (although for our practical numerical work we only need this structure at the origin) so our data would be invariant under $O'(3) \times T$. Equation (3.15) is satisfied if $\psi_0 = -\theta_0$, $B_0 = \pm\psi_0$, $\psi_0 = -3\phi_0$, $\mu_0{}^3 = \phi_0$. The particular numbers we have chosen for our computer work are $B_0 = 1$, $\theta_0 = -1$, $\psi_0 = +1$, $\phi_0 = -\frac{1}{3}$. With these choices we found, after an $e_{\alpha i}$ transformation, that (3.16) was zero to computer accuracy. $e_{\alpha i}$ is made orthogonal by requiring

$$\begin{aligned}
e_{11} &= l\lambda - p\pi + m\mu - n\nu \\
e_{12} &= l\pi + p\lambda + n\mu + m\nu \\
e_{13} &= -l\nu + p\mu - n\lambda + m\pi \\
e_{10} &= -l\mu - p\nu + n\pi + m\lambda \\
e_{21} &= -p\lambda - \pi l + m\nu + n\mu \\
e_{22} &= -p\pi + l\lambda - m\mu + n\nu \\
e_{23} &= p\nu + l\mu + m\lambda + n\pi \\
e_{20} &= p\mu - l\nu - m\pi + n\lambda \\
e_{31} &= n\lambda + m\pi + l\nu + p\mu \\
e_{32} &= n\pi - m\lambda - \mu l + p\nu \\
e_{33} &= -n\nu - m\mu + l\lambda + p\pi \\
e_{30} &= -n\mu + m\nu - l\pi + p\lambda \\
e_{01} &= -m\lambda + n\pi - \nu p + \mu l \\
e_{02} &= -m\pi - n\lambda + p\mu + l\nu \\
e_{03} &= m\nu - n\mu - p\lambda + l\pi \\
e_{00} &= m\mu + n\nu + p\pi + l\lambda
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned} p &= \sqrt{(1 - l^2 - m^2 - n^2)} \\ \pi &= \sqrt{(1 - \lambda^2 - \mu^2 - \nu^2)} \end{aligned} \tag{4.5}$$

Then we get

$$\begin{aligned} e_{\alpha i} e_{\alpha j} &= \delta_{ij} \\ e_{\alpha i} e_{\beta i} &= \delta_{\alpha\beta} \end{aligned} \tag{4.6}$$

Equation (4.6) was checked on the computer for all our choices of parameters. The parameters that we finally chose to work with are

$$l = \cdot 7 \qquad m = \cdot 4 \qquad n = \cdot 3 \qquad \lambda = \cdot 1 \qquad \mu = \cdot 5 \qquad \nu = \cdot 3 \tag{4.7}$$

We also took $g_{ij} = (1, 1, 1, 0)$ at the origin so as to make use of our g_{00} minimum equations. This set of data then leads to a minimum in g_{00} at the origin point.

In our previous computer runs in $g \neq 0$ theory, a general transformation led to situations where all sixty-four Γ^i_{jk} were different. In the present case, after an $e_{\alpha i}$ transformation, we see this is not the case. There are only eight different Λ_{ijk}. All the remaining Λ_{ijk} are repeats. We can take the different Λ_{ijk} to be Λ_{111}, Λ_{121}, Λ_{131}, Λ_{101}, Λ_{112}, Λ_{122}, Λ_{113}, Λ_{110}. Furthermore, some of these Λ_{ijk} are constant multiples of other members of the set. This has been demonstrated to computer accuracy, after running down the different axes. It has been found that $3\Lambda_{112} = \Lambda_{121}$, $3\Lambda_{113} = \Lambda_{131}$, $3\Lambda_{110} = \Lambda_{101}$, $\Lambda_{122} = 3\Lambda_{111}$. Thus, out of the sixty-four Λ_{ijk}, there are four independent components. This result is a consequence of our choice of $\Lambda_{\alpha\beta\gamma}$. When we took the example of Λ_{123}, Λ_{120}, Λ_{213}, Λ_{210}, Λ_{333}, Λ_{000} as the only non-vanishing $\Lambda_{\alpha\beta\gamma}$ components, we got sixty-four distinct Λ_{ijk} after the same $e_{\alpha i}$ transformation.

Is this diminishing of independent components an asset or liability? This is difficult to say. It may be that sixty-four pieces of information at the origin is too much. The spirit of this present investigation is to seek out possibilities consistent with the notion that the change function determines its own change. The number of parameters to be specified at the origin is not something that we can make definitive statements about at this time.

5. *Discussion*

In our work on aesthetic field theory, thus far, we have brought up (with varying degrees of discussion) several possibilities consistent with the program that the change function determines its own change. They are:

(a) $R^i{}_{jkl} = 0$ or $R^i{}_{jkl} \neq 0$.
(b) g positive, negative or zero.
(c) Symmetry restrictions on the change function, such as $\Lambda_{ijk} = -\Lambda_{jik}$.
(d) Higher dimensions (Muraskin, 1972b).
(e) Number of pieces of independent data that have to be supplied at the origin.
(f) Introduction of non-dynamical concepts such as metric and dimension.

These different possibilities when characterized by an exact set of data at the origin point, serve to define a universe, whether it is correct or not. If we could find a universe in which interesting things happen, this would be our immediate aim. We have already found in our previous work, some universes to be more interesting than others. For example, we presented a choice of data in Muraskin & Ring (1972a) which led to a minimum in g_{00}. Other data used in Muraskin (1972b) did not lead to a maximum (minimum) in g_{00} anywhere. There is hope that we might find a set of data for which a bounded particle exists. If, in the future, we ever found more than one particle, then we could see how they move with respect to one another enabling us to investigate force laws. These force laws could be compared with phenomena in the real world. The aesthetic field theory with all its distinct possibilities offers us a potential laboratory of universes which we can study with the hope of uncovering a world containing reasonable particle-like behavior.

At the moment, we have no iron-clad criterion to tell us which possibility mentioned above is most likely. Nor do we know how to choose the data so as to get optimum results for each of the possibilities. Nevertheless, at the moment, we see no other practical program than to try out different things and see what can be learned. We would like to systematically try out all the possibilities with the computer. However, the difficulty is that it is not a simple matter to find solutions to the integrability equations which are 444 non-linear algebraic equations in the $R^i{}_{jkl} \neq 0$ situation and 96 non-linear equations when $R^i{}_{jkl} = 0$ ($g \neq 0$ in both instances). We also have an infinite number of boundary restrictions when $g \neq 0$. We have found in previous papers that there exists solutions to the integrability equation such that all invariants involving Γ^i_{jk} are zero at infinity. However, in no case have we found any acceptable solutions for one reason or another.

In the $g = 0$ situation there are an infinite number of integrability equations that have to be satisfied. Our group theoretic data satisfied these equations to computer accuracy. We also have a minimum in g_{00} at the origin. We next discuss long-time runs on the computer in which we looked for a bound to the particle-like behavior.

6. *Computer Results*

Our data (4.1), (4.2) and (4.3) has led to an improvement over the results in previous papers. In our previous long-time computer run (Muraskin & Ring, 1972a), we found a monotonic change of all field components throughout the entire run. That is, those components that started getting bigger (smaller) continued to do so at an ever-increasing rate. Thus, no bound was found on the particle-like structure. Note, the data used in this case can be shown to violate the boundary condition $\Gamma^i_{jk} \to 0$ at infinity.

On the other hand, a long-time run down the x-axis using our data (4.1), (4.2) and (4.3) did lead to a bound on our particle. At $x = 0$, we had $g_{00} = 0$. This increased to ·273 (rounded off) at $x = 2{\cdot}178$. We used a grid size of ·0003 in reaching this point. This took about 7 hours of running time to reach. After this point, g_{00} started to decrease with a slow rate, and continued to do so throughout the rest of the run. At $x = 2{\cdot}97$, g_{00} had the value ·264. At $x = 4{\cdot}77$ it had the value ·230. We note also that Λ_{111}, which started off at $-{\cdot}133$, decreased to $-{\cdot}181$ at $x = {\cdot}45$, and then started to increase again. Thus, a second new feature (in addition to the bound on the g_{00} particle) is that Λ_{ijk} components were not monotonic. In fact, over the x-direction width of the particle, $-1{\cdot}38 < x < 2{\cdot}178$, Λ_{111} had two turnabout points. A_{121}, A_{131}, A_{101} each had one turnabout point. Finally another interesting result emerged. Outside the particle all A_{ijk} tended monotonically towards zero. At $x = 0$ we had

$$A_{111} = -\cdot 133$$
$$A_{121} = -\cdot 122$$
$$A_{131} = -\cdot 483$$
$$A_{101} = -\cdot 771$$

At $x = 4{\cdot}77$ the values of these quantities were (we used progressively larger grids as the fields got smaller)

$$A_{111} = -\cdot 0625$$
$$A_{121} = -\cdot 0055$$
$$A_{131} = -\cdot 0175$$
$$A_{101} = -\cdot 0280$$

Next, we ran down the other axes. We found a bound in each direction. The turnabout point for g_{00} and the value of g_{00} at the turnabout point are given below

$x = 2{\cdot}178$	$g_{00} = \cdot 273$
$x = -1{\cdot}38$	$g_{00} = \cdot 729$
$y = 1{\cdot}854$	$g_{00} = \cdot 413$
$y = -1{\cdot}626$	$g_{00} = \cdot 536$
$z = 2{\cdot}28$	$g_{00} = \cdot 233$
$z = -1{\cdot}314$	$g_{00} = \cdot 802$

The qualitative results for these axes were similar to the results for the x-axis. Our longest run was down the x-axis (see above).

All A_{ijk} decreased in magnitude along the x_0-axis. Along the $-x_0$-axis all A_{ijk} began to increase in magnitude but eventually they began to decrease.

We also at $x = 4{\cdot}77$ ran along the y-axis to $y = 2{\cdot}4$. A_{111}, A_{131}, A_{101} were smaller in magnitude than what they were at $x = 4{\cdot}77$. A_{121} was bigger in magnitude. But its rate of increase was already slowing down at this point.

Thus, all the evidence points to a localized particle object associated with g_{00} surrounded by a 'vacuum' where the A_{ijk} get small.

At this point we cannot say whether we have a universe with one particle in it, and in which A_{ijk} tends to zero at infinity or whether more structure will develop farther from the origin. The fact that A_{ijk} gets smaller as we go farther from the origin does not mean that A_{ijk} will continue in this fashion with still longer computer runs. That is, a set of small A_{ijk} need not get smaller. To illustrate this, if we run backwards towards the origin the small A_{ijk} will get larger.

Even if there were but a single particle in our universe, the results would still be remarkable in the following sense. A problem with $g = 0$ field theory is that we have no handle on the behavior of A_{ijk} at infinity. When $g \neq 0$ we can at least require all Γ^i_{jk} invariants be zero at the origin which is necessary for $\Gamma^i_{jk} \to 0$ at infinity. But in $g = 0$ theory, we do not have this handle on the boundary conditions since invariants are not zero any more. A possible interpretation we can extract from our computer work is that A_{ijk} tends to zero at infinity without the need of imposing additional conditions on the theory.

We can say that our particle is as good as that of Rosen (1966), Born & Infeld (1934) or Anderson & Derrick (1970). Our more structured particle differs from theirs in that it is not spherically symmetric. It should also be noted that our particle emerges out of a mathematically aesthetic program rather than from *ad hoc* field equations. Also there is a possibility that an additional structure will appear with longer-time computer runs.

We have inferred the existence of a bounded particle-like object from computer runs along the coordinate axes. Strictly speaking, we should map out regions surrounding the origin. Limitations on computer time are a factor here, if we wish to maintain the kind of numerical accuracy we have been getting up to now.†

7. *Discussion*

In order to obtain the computer results we need only to assume the data given by (4.1), (4.2), (4.3) and (3.14) at one point and then we generated the field at all points using the field equations (3.4) and (3.5). In view of the rather remarkable numerical results we have obtained, we may ask whether there are basic principles at work. With this in mind, we have made the hypothesis that the underlying structure (characterized by $g_{\alpha\beta}$, $A_{\alpha\beta\gamma}$) is invariant under $O'(3) \times T$.

We have supposed that $A_{\alpha\beta\gamma}$ is constant. Then, in order for $A_{ijk} \to 0$ at infinity we must require $e_{\alpha i} \to 0$ at infinity. We have from (3.2)

$$\frac{\partial e_{\alpha i}}{\partial x_k} = A_{mik}\, e_{\alpha m} \tag{7.1}$$

We have taken (4.6) to hold at the origin. We can now see that it does not hold at all points. From (7.1) we get at the origin where we may use (4.6)

$$\frac{\partial(e_{\alpha i}\, e_{\alpha j})}{\partial x_k} = A_{jik} + A_{ijk} \tag{7.2}$$

Our data (4.1), (4.2) and (4.3) does not satisfy $A_{ijk} = -A_{jik}$. Thus, (7.2) is not zero. Similarly we get

$$\frac{\partial(e_{\alpha j}\, e_{\beta j})}{\partial x_k} = e_{\beta i}\, e_{\alpha m}(A_{mik} + A_{imk}) \tag{7.3}$$

The fact that (7.2) and (7.3) are not zero is essential. Otherwise we would get an inconsistency with the boundary condition $A_{ijk} \to 0$ at infinity. This is because from

$$\begin{aligned} e_{\alpha i}\, e_{\alpha j} &= \delta_{ij} \\ e_{\alpha i}\, e_{\beta i} &= \delta_{\alpha\beta} \end{aligned} \tag{7.4}$$

† We did, however, make a run to the point $x = 4{\cdot}77$, $y = 2{\cdot}4$ as discussed previously.

we get $\det e_{\alpha i} \equiv e = 1$, and thus $e_{\alpha i}$ could not go to zero at infinity and still have determinant one.

In Section 3 we discussed the $e_{\alpha i} \neq 0$ at infinity $g = 0$ situation. This present section examines $g = 0$ when $\Lambda_{\alpha\beta\gamma}$ has the structure given by (4.1), (4.2) and (4.3). We can relook at one of the problems that came up before we introduced (4.1), (4.2) and (4.3). We remember that contracted quantities have a different sort of change equation than we might naively suspect. For example, $c_k = g_{im}\Lambda_{imk}$ does not act like a vector with respect to its change. This is the way it should be. We may see this if we express everything in terms of $e_{\alpha i}$. We then get for the change of c_k the following

$$d(e_{\alpha i} e_{\beta m} g_{\alpha\beta} e_{\sigma i} e_{\rho m} e_{\lambda k} \Lambda_{\sigma\rho\lambda}) \tag{7.5}$$

But this is not the same as

$$d(e_{\gamma k} g_{\alpha\beta} \Lambda_{\alpha\beta\gamma}) \tag{7.6}$$

since $e_{\alpha i} e_{\sigma i}$ is not a constant.†

In the following, we shall summarize our basic approach. We have introduced a change function that describes the change of a basis vector set of fields $e_{\alpha i}$ according to

$$de_{\alpha l} = \Lambda_{mlk} e_{\alpha m} dx_k \tag{7.7}$$

The simplest way we can express the Λ_{ijk} in terms of the basis vectors is

$$\Lambda_{ijk} = e_{\alpha i} e_{\beta j} e_{\gamma k} \Lambda_{\alpha\beta\gamma} \tag{7.8}$$

with $\Lambda_{\alpha\beta\gamma}$ constant. We next require that $\Lambda_{\alpha\beta\gamma}$ be invariant under the transformation $O'(3) \times T$. A $\Lambda_{\alpha\beta\gamma}$ that does this, and is constructed from a $g_{\alpha\beta}$ having determinant zero, is given by (4.1), (4.2) and (4.3). We require that integrability be satisfied. Next, from the quantities μ_α, ϕ_α, θ_α, ψ_α, B_α, $g_{\alpha\beta}$ appearing in (4.1), (4.2) and (4.3) we can form the quantities

$$\begin{aligned} \phi_i &= e_{\alpha i} \phi_\alpha \\ g_{ij} &= e_{\alpha i} e_{\beta j} g_{\alpha\beta} \qquad \text{etc.} \end{aligned} \tag{7.9}$$

Then, the change of these quantities is given by using (7.7). From (7.7), (7.8) and (7.9) we get the equations (3.2), (3.4) and (3.5). In order to satisfy the boundary conditions $\Lambda_{ijk} \to 0$ at infinity $e_{\alpha i} \to 0$ at infinity is required. This implies that not all vectors behave in the same way so far as their change is concerned (see (7.5) and (7.6)). Thus, we need a different principle than in our previous papers. We require that the change function determines the change of all functions in a manner depending on the way that the functions are constructed from $e_{\alpha i}$. The change function must determine itself by this same principle. This gives rise to the field equations. According to the rule, quantities like $e_{\alpha i} e_{\alpha j}$ have their change given by (7.2), etc. The group theoretic argument is of great importance since other $\Lambda_{\alpha\beta\gamma}$ that we have tried in Section 3 all led to obvious difficulties.

We have thus constructed a simple aesthetic framework that leads to the field equations used in the computer program and which uses the same data at the origin for Λ_{ijk}.

† The non-dynamic metric introduced earlier is preserved by orthogonal transformations. In this section $e_{\alpha i}$ is not orthogonal so we shall not introduce such a concept here.

8. *Conclusions*

It has become clear to us that aesthetic field equations are, by themselves, not sufficient. We have found the solutions are just too dependent on the origin point data. It is therefore necessary to have 'aesthetic' origin point data. The behavior at infinity has an important role in determining the data at the origin. For example, the sign of g is preserved by the field equations. Thus, the invariance group at infinity can be used to determine the sign of g at the origin point. We have made the hypothesis that this invariance group is $O'(3) \times T$. This led us to consider the case $g = 0$. However, $g = 0$ is by itself an insufficient principle as we found several sets of $\Lambda_{\alpha\beta\gamma}$ within $g = 0$ theory that were obviously unacceptable. This led us to the far-reaching hypothesis that there exists an underlying group structure that can be reached through an $e_{\alpha i}$ transformation. The group was taken again to be $O'(3) \times T$.† Such a hypothesis represents an attempt at prescribing the data in an aesthetic manner.

From the field equations (3.4) and (3.5) and the data (4.1), (4.2), (4.3) and (3.14) we find the following results:

(a) Bounded particle behavior (as determined by runs down the coordinate axes, as well as a run off the axis to the point $x = 4{\cdot}77$, $y = 2{\cdot}4$).
(b) Absence of any trends toward singularities.
(c) Λ_{ijk} becoming very small in magnitude outside the particle (this suggests that the natural boundary conditions at infinity $\Lambda_{ijk} \to 0$ are not unlikely).

None of these results have been obtained by us previously. This tends to confirm the feeling that our present data represent an important step forward in our problem.

A possibility exists that longer runs on the computer are necessary before still additional particle structures begin to show up. Another possible inference is that our universe has but one particle in it and thus we have not been aesthetic enough.

Acknowledgement

The author would like to thank Beatrice Ring for her very able computer programming. The author also thanks Jarvis Ehart for operating the computer. A research grant from the University of North Dakota and generous allowance of computer time by the university is also gratefully acknowledged.

References

Anderson, D. L. T. and Derrick, G. H. (1970). *Journal of Mathematical Physics*, **11**, 1336.
Anderson, J. L. (1967). *Principles of Relativity Physics*. Academic Press, Chapter 5.
Born, M. and Infeld, L. (1934). *Proceedings of the Royal Society*, **A144**, 425.

† We are assuming that the only coordinate transformations on $\Lambda_{\alpha\beta\gamma}$ and $g_{\alpha\beta}$ that are not equivalent to a dynamical effect are translations of the origin and rotations of the space axes.

Lanzcos, C. (1966). *Journal of Mathematical Physics*, **7**, 316.
Muraskin, M. (1970). *Annals of Physics*, **59**, 27.
Muraskin, M. (1971a). *Journal of Mathematical Physics*, **12**, 28.
Muraskin, M. (1971b). *International Journal of Theoretical Physics*, Vol. 4, No. 1, p. 49.
Muraskin, M. (1972a). *Foundations of Physics*, **2**, 181.
Muraskin, M. (1972b). *International Journal of Theoretical Physics*, Vol. 6, No. 1, p. 37.
Muraskin, M. (1972c). *Journal of Mathematical Physics*, **13**, 863.
Muraskin, M. and Clark, T. (1970). *Annals of Physics*, **59**, 19.
Muraskin, M. and Ring, B. (1972a). *International Journal of Theoretical Physics*, Vol. 6, No. 2, p. 105.
Muraskin, M. and Ring, B. (1972b). Preprint.
Rosen, G. (1966). *Journal of Mathematical Physics*, **7**, 2066.
Trautmann, A. (1964). *Lectures in General Relativity*, Vol. I, *Brandeis Lectures 1964*. Prentice Hall.

Note in Proof

We would like to draw attention to an errata to be found in *Annals of Physics* **61**, 260 (1970).

In Muraskin & Ring (1972a) the grid size used was 0·0001 and not 0·001.

In Muraskin (1972a) the last part of equation (18) should read $A^1_{21} = -A^2_{11}$.

In Muraskin (1972b) delete as misleading the sentence immediately following equation (2.8). In the same paper, the footnote on page 44 should read '... we do have $\Gamma^i_{jk} \to 0$ at spatial and temporal infinity for certain directions from the origin.' Also, in the second sentence of the last paragraph on page 44, the word 'would' should appear as 'could'.